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LOWER BOUNDS TO LARGE DISPLACEMENTS OF IMPULSIVELY LOADED PLASTICALLY ORTHOTROPIC STRUCTURES

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Abstract—Impulsively loaded plastic structures deform beyond the limits of applicability of the geometrically linear theory. It was experimentally observed that due to the membrane action actual permanent displacements are smaller than those predicted by the infinitesimal theory. Exact solutions for deformed shapes in the geometrically nonlinear range are not known for anisotropic structures.

The note advances a technique allowing to bound from below the permanent, moderately large deflection at a chosen point of a rigid-plastic, dynamically loaded structure. The method originally developed for isotropic solids and introducing an auxiliary kinematically admissible velocity field allowing to estimate the dissipation due to the nonlinear terms in the strain rates is extended to orthotropic plates and shells.

Lower bounds are obtained to maximum deflections of circular orthotropic plates obeying a piece-wise linear yield criterion when accounting for moderately large displacements. The influence of orthotropy on the permanent deflections is discussed and the results are compared to those of the linear theory. Meaningful differences are noticed, particularly for more intense impulses. Results for a cylindrical shell are also presented.

1. INTRODUCTION

Impulsively loaded plastic structures may deform significantly during the process of motion. An analysis of permanent deformations is therefore needed in the geometrically nonlinear range. Most of the attention related to plastic behavior of beams, plates and shells concentrated on small displacements[1–3]. For moderately large deflection theories exact solutions, regarding the deformed shape at rest after impact, exist for beams and circular plates[4, 5]. Experiments are reported in [6].

As the exact solutions in the plasto-dynamics are difficult to obtain a particular attention was given to bounding techniques allowing to estimate the time of motion and the final maximum deflections[7–12]. Techniques were developed as well regarding upper bounds to deflections in the nonlinear range[13–15]. Recently a method of assessing the permanent deflection from below has been proposed and successfully applied to structures made of isotropic, perfectly plastic materials[16–17]. Lower bound techniques in the nonlinear range involve some information regarding the deformation itself as the stress and deformation are coupled via the governing equations. Hence a lower bound technique in the nonlinear deformation range imposes the requirements not appearing at small displacements when the deformed and undeformed configurations can be referred to the material co-ordinate system without any essential modification.

On the other hand plastic anisotropy influences the behavior of structures. The differences consist not solely in value of the collapse load[18–19], but concern the collapse modes as well. Hence it appears worthwhile to study the behavior of plastically anisotropic structures in the dynamic range at moderately large deflections. In the linear range the problem was studied in [20–21].

The present note concerns estimations of the permanent deflections of rigid, perfectly plastic structures. A lower bound technique exposed in [16] is extended on anisotropic solids

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and structures in order to estimate both the influence of anisotropy and of moderately large deflections on the final displacements of impulsively loaded plates and shells. The material is assumed to be incompressible and following the piecewise linear yield criterion in the space of principal stresses[22].

In Section 2 the bounding technique developed in [17] is recalled with unessential modifications as required by the employed criterion of yielding. The yield condition is specified next in application to circular plates and cylindrical shells exhibiting a plastic anisotropy. Section 4 gives estimations for orthotropic plates, studies the influence of the orthotropy ratio on the permanent deflections and gives comparisons with the isotropic case. Cylindrical shells of a sandwich wall are considered in the next section. Section 6 contains conclusions and general remarks as justified by the analysis performed.

2. LOWER BOUNDS TO LARGE DISPLACEMENTS

Let us consider a rigid-plastic solid of volume V_0 and surface S_0 in the initial configuration. The body is referred to a cartesian coordinate system x_i associated with the initial configuration. The material reference system will be employed throughout. The original smooth surface S_0 consists of the part S_{0p} where the surface tractions $P_i(\mathbf{x}\, t)$ are prescribed and of the part S_{0u} where the displacements U_i^S or velocities $\dot{U}_i^S$ are given. The surface tractions are considered conservative throughout the deformation process. The mass per unit volume is denoted by ρ.

In the solid in motion we consider two fields of displacements, velocities and accelerations satisfying the prescribed kinematical constraints. The fields $U_i(\mathbf{x}, t)$, $\dot{U}_i(\mathbf{x}, t)$ and $\ddot{U}_i(\mathbf{x}, t)$ denote the exact solution whereas $U_i^*(\mathbf{x}, t)$, $\dot{U}_i^*(\mathbf{x}, t)$ and $\ddot{U}^*(\mathbf{x}, t)$ represent kinematically admissable vector fields of displacements, velocities and accelerations, respectively. The problem setting is thus as that employed in [16, 17] for isotropic solids. To make the note selfcontained we develop the basic equations of the adopted approach, adding the modifications imposed by the requirement of plastic anisotropy. Equations of the large deflection theory of shells are exposed in [23]. The equations of motions are

$$[(\delta_{ij} + U_{i,j})T_{kj}]_{,k} = \rho \ddot{U}_i \tag{2.1}$$

where T_{ij} is the symmetric Piola–Kirchhoff stress tensor. The stress field is subjected to the boundary conditions

$$P_i = [(\delta_{ij} + U_{i,j})T_{kj}]N_k \quad \text{on } S_{ip} \tag{2.2}$$

N_k standing for components of the normal vector to S_{0p}.

The displacement and velocity boundary conditions are respectively for the exact U_i and a kinematically admissible U_i^* fields as follows

$$U_i(\mathbf{x}, t) = U_i^S \quad \text{and} \quad \dot{U}_i(\mathbf{x}, t) = \dot{U}_i^S \quad \text{on } S_{0u} \tag{2.3}$$

$$\dot{U}_i^*(\mathbf{x}, t) = \dot{U}_i^S \quad \text{on } S_{0u}. \tag{2.4}$$

The initial conditions specify the requirements imposed on the displacement and velocity fields at the application of impulsive loading. For the exact solution they are

$$U_i(\mathbf{x}, 0) = 0, \; \dot{U}_i(\mathbf{x}, 0) = \dot{U}_{0i} \quad \forall \mathbf{x} \in V_0. \tag{2.5}$$

For an impulsively loaded structure the motion following the exact velocity field ceases at $t = t_f$. When we consider a kinematically admissible velocity field $\dot{U}_i^*$ the motion might cease at another instant of time $t = t^*$. Hence the final conditions are

$$\dot{U}(\mathbf{x}, t_f) = 0, \; \dot{U}^*(\mathbf{x}, t^*) = 0 \quad \forall \mathbf{x} \in V_0. \tag{2.6}$$

The Green strain rate tensor specifying the deformation rate field considered continuous in

V_0, is

$$\dot{E}_{ij} = \tfrac{1}{2}(\dot{U}_{i,j} + \dot{U}_{j,i} + \dot{U}_{k,i}U_{k,j} + \dot{U}_{k,j}U_{k,i}). \tag{2.7}$$

For completeness we specify a yield condition. It was shown in [24] that at moderately large deflections of plates and shells the yield condition written in terms of the Cauchy stress can be assumed in the same form if written in the initial configuration in terms of the symmetric Piola–Kirchhoff stress. In the case of anisotropic materials the parameters A_i, say, describing directional properties of the solid enter the criterion. Thus

$$\Phi(T_{ij}, A_i) = 0 \tag{2.8}$$

is the general form of the yield condition which we shall refer to the preferred directions of anisotropy.

It is assumed that the plastic flow law applies thus the rates (2.7) during plastic flow are as follows

$$\dot{E}_{ij} = \lambda \frac{\partial \Phi}{\partial T_{ij}}, \quad \lambda \geqslant 0. \tag{2.9}$$

Having recalled the basic relations and conditions we can proceed to establish a bound to the displacements at moderately large geometry changes. To this end we employ the principle of virtual work considering the state in equilibrium on a virtual field of displacement rates $\dot{U}^*_i$. Multiplying (2.1) by $\dot{U}^*_i$ and employing the stress boundary condition (2.2) one eventually obtains

$$\int_{S_{0p}} P_i\dot{U}^*_i\,\mathrm{d}s - \int_{V_0} \rho\ddot{U}_i\dot{U}^*_i\,\mathrm{d}V = \int_{V_0} (\delta_{ij} + U_{i,j})T_{kj}\dot{U}^*_{i,k}\,\mathrm{d}V. \tag{2.10}$$

The r.h.s. integral in (2.10) represents the rate of internal work on the kinematically admissible velocity field $\dot{U}^*_i$. The above expression can thus be expressed as follows

$$\int_{S_{0p}} P_i\dot{U}^*_i\,\mathrm{d}s = \int_{V_0} \rho\ddot{U}_i\dot{U}^*_i\,\mathrm{d}V = \int_{V_0} T_{ij}\dot{E}^*_{ij}\,\mathrm{d}V + \int_{V_0} T_{ij}\dot{E}^{**}_{ij}\,\mathrm{d}V \tag{2.11}$$

where

$$\dot{E}^{**}_{ij} = \tfrac{1}{2}(\dot{U}^*_{i,j} + \dot{U}^*_{j,i}), \quad \dot{E}^{**}_{ij} = \tfrac{1}{2}(\dot{U}^*_{k,i}U_{k,j} + \dot{U}^*_{k,j}U_{k,i}) \tag{2.12}$$

are respectively linear and nonlinear parts of the strain rate. The nonlinear part involves both the exact displacement field U_i as well as the kinematically admissible velocity field $\dot{U}^*_i$. As it is seen from (2.12) the expression (2.11) contains the displacement field of the unknown exact solution. Our goal is now to estimate the exact displacement field, or at least to get a bound to the most interesting from the engineering point of view, components of moderately large displacement vector U_i.

We shall now pass to an estimation of the left hand side of (2.11). To this end we consider for the time being $\dot{E}^*_{ij}$, and $\dot{E}^{**}_{ij}$ as two independent kinematically admissible strain rate fields derived from two kinematically admissible fields U_i and $\dot{U}^*_i$. This separation of the strain rate (2.7) into two parts defined in (2.11) is essential for the proposed bounding technique. Instead of calculating the right hand side of (2.11) we shall establish a bound to its value making use of the Drucker postulate. Employing the plastic potential flow law to the yield criterion (2.8) we can find the corresponding values T^*_{ij} and T^{**}_{ij} since the dissipation is fully determined by the kinematical requirements once the yield condition is specified. The components T^*_{ij} and T^{**}_{ij} follow directly from the plastic potential flow law once the separation (2.12) is made. The yield condition convexity allows to arrive at an inequality for the internal dissipation of the kinematical admissible velocity field $\dot{U}_i^*$. Instead of (2.11) one obtains

$$\int_{S_{0p}} P_i \dot{U}_i^* \, ds - \int_{V_0} \rho \ddot{U}_i \dot{U}_i^* \, dV \leq \int_{V_0} T_{ij}^* \dot{E}_{ij}^* \, dV + \int_{V_0} T_{ij}^{**} \dot{E}_{ij}^{**} \, dV. \tag{2.13}$$

This inequality is the basic one for an estimation of a bound to the displacement U_i, which is the only unknown quantity in (2.13).

Let us integrate (2.13) within the time interval $(0, t^*)$, thus from the beginning up to the termination of motion. The integration yields

$$\int_0^{t^*} \int_{S_{0p}} P_i \dot{U}_i^* \, dV \, dt + \int_{V_0} \rho \dot{U}_{0i} \dot{U}_{0i}^* \, dV - \int_0^{t^*} \int_{V_0} T_{ij}^* \dot{E}_{ij}^* \, dV \, dt \leq$$

$$- \int_{V_0} \rho U_i(t^*) \ddot{U}_i(t^*) \, dV + \int_0^{t^*} \int_{V_0} \rho U_i \ddot{U}_i^* \, dV \, dt + \int_0^{t^*} \int_{V_0} \dot{E}_{ij}^{**} T_{ij}^{**} \, dV \, dt. \tag{2.14}$$

It has to be remarked that the initial conditions (2.5), which are known and apply both to $\dot{U}_i$ and $\dot{U}_i^*$, enter the inequality.

When a kinematically admissible velocity field $\dot{U}_i^*$ is chosen the l.h.s. of (2.14) contains solely known quantities. The r.h.s. involves the only one unknown, namely U_i. Hence (2.14) can serve to estimate this displacement. Its value depends on the choice of a kinematically admissible velocity field $\dot{U}_i^*$. Moreover, since $\dot{U}_i^*$ can be chosen arbitrarily among kinematical admissible fields, and independently from the time of actual motion t_f, the time t^* can be arbitrary. We shall proceed in selecting t^* so as to obtain the best bound to U_i in the considered class of $\dot{U}_i^*$. For our purpose it is immaterial whether t^* is smaller or larger in comparison with t_f since we are looking for an estimate of U_i, not for its exact value.

The last integral in (2.14) involves U_i as it is seen from (2.12). Employing the bounding principle for dissipation as proposed in [25] one obtains

$$\int_{V_0} T_{ij}^{**} \dot{E}_{ij}^{**} \, dV \leq \kappa \int_{V_0} \| U_{k,i} \dot{U}_{k,j}^* + U_{k,j} \dot{U}_{k,i}^* \| \, dV \tag{2.15}$$

where κ is specified as follows.

$$\kappa = S_{up} \frac{T_{ij}^{**} \dot{E}_{ij}^{**}}{\| \dot{E}_{ij}^{**} \|}, \quad \| \dot{E}_{ij}^{**} \| = \sqrt{(\dot{E}_{ij}^{**} \dot{E}_{ij}^{**})} \tag{2.16}$$

and depends on the yield condition. Its magnitude for the employed yield criteria will be given when the criterion used is discussed.

It is not attempted to estimate the field $U_i(x, t)$ but the attention is focussed on bounding of a specific component of the displacement, at an arbitrary point of the structure. This component k, say, might be chosen at a specific point. For plates it will be, e.g. an estimation of the maximum deflection. A kinematically admissible displacement rate field of the type

$$\dot{U}_i^*(\mathbf{x}, t) = \delta_i^k a(\mathbf{x}, t), \quad k \text{ specified} \tag{2.17}$$

where $a(\mathbf{x}, t)$ is usually chosen in a modal form

$$a(\mathbf{x}, t) = \frac{t^* - t}{t^*} A(\mathbf{x}). \tag{2.18}$$

We restrict our attention to modal forms of velocity fields, both for clarity of exposition and no need for optimisation. Once a kinematically admissible velocity field (2.17) is chosen an essential point is to estimate the dissipation due to the nonlinear part as stated in (2.15). The integration of (2.15) is needed. If only one component of U_i is in question, one arrives eventually at an estimation

$$\int_0^{t^*}\int_{V_0} T_{ij}^{**}\dot{E}_{ij}^{**}\,\mathrm{d}V\,\mathrm{d}t \leq C U_{\max}^{k} \tag{2.19}$$

where C is a constant to be determined depending on the kinematically admissible displacement rate field (2.17) chosen and the yield condition employed[16].

The bounding principle for the chosen component of the displacement can now be written, employing (2.19) in (2.14). Imposing some restrictions as to the continuity of the kinematically admissible velocity field $\dot{U}_i^*$ and making use of the bounds to the integrals in (2.14), the final result is

$$U_{\max}^{k} \geq \frac{\int_0^{t^*}\int_{S_{op}} P_i\dot{U}_i^*\,\mathrm{d}s\,\mathrm{d}t + \int_{V_0}\rho\dot{U}_{0i}\dot{U}_{0i}^*\,\mathrm{d}V - \int_0^{t^*}\int_{v_0} T_{ij}^*\dot{E}_{ij}^*\,\mathrm{d}V\,\mathrm{d}t}{-\int_{V_0}\rho\dot{U}^{k*}\,\mathrm{d}V + \int_0^{t^*}\int_{V_0}\rho\ddot{U}^{k*}\,\mathrm{d}V\,\mathrm{d}t + C} \tag{2.20}$$

and the best value is obtained selecting t^* so that the r.h.s. in (2.20) attains its maximum.

The presented principle of arriving at a lower bound to specific displacement will now be applied to anisotropic plates and shells. It was previously used to isotropic structures in [16, 17]. The formula (2.20) will be given for moderately large deflection estimation. The integrations prescribed in (2.20) will then be performed and the constant appearing in (2.19), evaluated. The method of bounding from below supplements the available techniques of upper bound evaluations when an auxiliary solution for a point loaded structure is needed[5, 14].

3. ORTHOTROPIC YIELD CRITERION

We consider plastically orthotropic solids obeying a piece-wise linear criterion of yielding. For definiteness the principal directions of stress are assumed to coincide with the privileged material orientations. If T_1, T_2, T_3 denote the principal values of stress and Y_1, Y_2, Y_3 the respective values of yield stress the yield criterion proposed in [22] for hydrostatic pressure insensitive materials has the form

$$\begin{aligned} &\frac{T_1-T_3}{Y_1}-\frac{T_2-T_3}{Y_2}=\pm 1\\ -&\frac{T_2-T_1}{Y_2}-\frac{T_3-T_1}{Y_3}=\pm 1\\ &\frac{T_1-T_2}{Y_1}-\frac{T_3-T_2}{Y_3}=\pm 1. \end{aligned} \tag{3.1}$$

To assure convexity of the yield locus the yield points have to satisfy the conditions explicited in [22].

For plates and shells it is necessary to establish the yield criteria in terms of the stress resultants, therefore to specify appropriate yield surfaces in the space of stress resultants. In the case of large deflections of circular plates the yield surface has to be expressed in terms of bending moments M_Θ, M_r and membrane forces N_Θ, N_r.

For the yield criterion (3.1) we consider the limited interaction yield locus as originally proposed in [26] for isotropic shells. If the plate wall thickness is 2H, Y is the reference yield stress and $Y_\Theta=\beta Y$, $Y_r=\gamma Y$. We introduce the reference yield moment $M_0=YH^2$ and yield axial force $N_0=2YH$. Dimensionless stress resultants are defined as follows

$$m_r=\frac{M_r}{M_0};\ m_\Theta=\frac{M_\Theta}{M_0};\ n_r=\frac{N_r}{N_0};\ n_\Theta=\frac{N_\Theta}{N_0}. \tag{3.2}$$

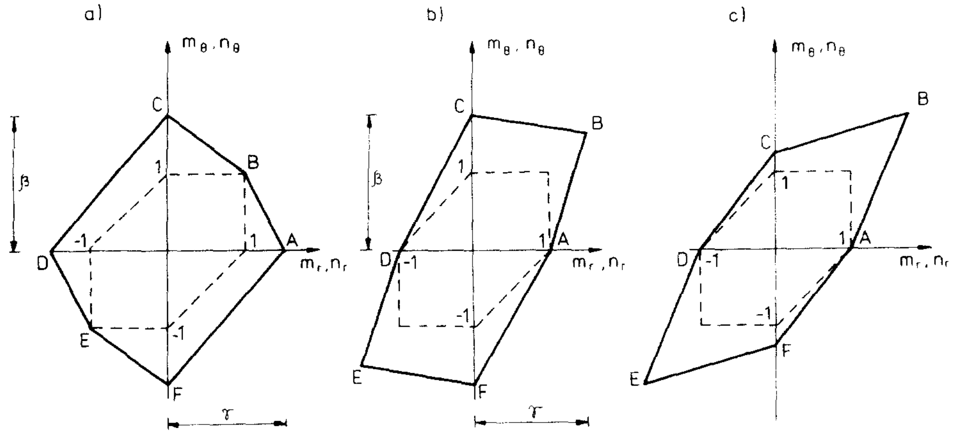

Fig. 1. Intersections of the limited interaction yield surfaces for orthotropic shells as well as for plates at large displacements.

The limited interaction yield surface imposes no interaction between the bending and membrane stress resultants in two principal directions. For the criterion stated in (3.1) the limited interaction yield surfaces can be visualized as in Fig. 1. The interactions take place solely in the moment and axial forces planes.

In Table 1 equations of the limited interaction yield surfaces are shown. The formulas apply under the conditions that $\beta > 1$ as well as $\beta \geq \gamma(1+\gamma)$. The first column of equations regards the case shown in Fig. 1(a) whereas the second concerns Figs. 1(b, c). The associated vectors of plastic strain rates are straightforward to obtain from the plastic potential flow law and will not be explicited here. It is clear that for a given set of kinematically admissible generalized strain rates q_i^* the associate set of generalized stresses Q_i^* is obtained employing the plastic potential flow law as it was already mentioned in Section 2 when a kinematically admissible velocity field was introduced.

For cylindrical shells at moderately large deflections the circumferential bending moment can be eliminated from the yield condition as the circumferential curvature change is neglected and therefore the circumferential moment does not contribute to the internal dissipation. The

Table 1. Limited interaction yield surfaces for rotationally symmetric orthotropic shells at various orthotropy ratios

	Yield equation	
1	$n_r + (\gamma - 1)n_\theta = \gamma$	$(\gamma - 1)n_\theta - \gamma n_r = \gamma$
2	$n_\theta + (\beta - 1)n_r = \beta$	$n_\theta - \frac{(\gamma - \beta)}{\gamma} n_r = \beta$
3	$n_r/\gamma - n_\theta/\beta = 1$	$n_\theta/\beta - n_r = 1$
4	$n_r + (\gamma - 1)n_\theta = -\gamma$	$(\gamma - 1)n_\theta - \gamma n_r = -\gamma$
5	$n_\theta + (\beta - 1)n_r = -\beta$	$n_\theta - \frac{(\gamma - \beta)}{\gamma} = -\beta$
6	$n_r/\gamma - n_\theta/\beta = -1$	$n_\theta/\beta - n_r = -1$
7	$m_r + (\gamma - 1)m_\theta = \gamma$	$(\gamma - 1)m_\theta - \gamma m_r = \gamma$
8	$m_\theta + (\beta - 1)m_r = \beta$	$m_\theta - \frac{(\gamma - \beta)}{\gamma} m_r = \beta$
9	$m_r/\gamma - m_\theta/\beta = 1$	$m_\theta/\beta - m_r = 1$
10	$m_r + (\gamma - 1)m_\theta = -\gamma$	$(\gamma - 1)m_\theta - \gamma m_r = \gamma$
11	$m_\theta + (\beta - 1)m_r = -\beta$	$m_\theta - \frac{(\gamma - \beta)}{\gamma} m_r = -\beta$
12	$m_r/\gamma - m_\theta/\beta = -1$	$m_\theta/\beta - m_r = -1$

Table 2. Yield surfaces for isotropic and orthotropic cylindrical shells

	Yield equation	
	isotropic	orthotropic
1	$-n_\varphi = 1$	$-n_x(1-\gamma) - \gamma n_\varphi = 1$
2	$n_\varphi = 1$	$n_x(1-\gamma) + \gamma n_\varphi = 1$
3	$n_x - 2n_\varphi + m_x = 2$	$\gamma n_x - 2\gamma n_\varphi - (2-\gamma)m_x = 2$
4	$-n_x + 2n_\varphi - m_x = 2$	$-\gamma n_x + 2\gamma n_\varphi - (2-\gamma)m_x = 2$
5	$n_x - 2n_\varphi - m_x = 2$	$\gamma n_x - 2\gamma n_\varphi - (2-\gamma)m_x = 2$
6	$-n_x + 2n_\varphi + m_x = 2$	$-\gamma n_x + 2\gamma n_\varphi + (2-\gamma)m_x = 2$
7	$n_x - n_\varphi = 1$	$n_x - \gamma n_\varphi = 1$
8	$-n_x + n_\varphi = 1$	$-n_x - \gamma n_\varphi = 1$
9	$n_x - m_x = 1$	$n_x - m_x = 1$
10	$-n_x + m_x = 1$	$-n_x - m_x = 1$
11	$-n_x - m_x = 1$	$-n_x - m_x = 1$
12	$n_x + m_x = 1$	$n_x - m_x = 1$

yield criterion (3.1) can be explicited as follows

$$\frac{Y_x - Y_\varphi}{Y_x Y_\varphi} T_x - \frac{T_\varphi}{Y_\varphi} = \pm 1$$

$$\frac{T_x}{Y} - \frac{Y_x - Y}{Y_x Y} T_\varphi = \pm 1 \tag{3.3}$$

$$\frac{T_x}{Y} - \frac{T_\varphi}{Y_\varphi} = \pm 1.$$

Considering sandwich shells and denoting $\gamma = Y_x/Y$, $\beta = Y_\varphi/Y$ under the requirement $\gamma \geq \beta/(1+\beta)$ one eventually obtains the set of yield planes as given in Table 2 for the case $\beta = 1$. For comparison the left side of Table 2 contains the relations applicable to isotropic cylindrical sandwich shells.

To complete the discussion of anisotropic yield criteria to be applied to the deflection evaluation according to (2.20) we have to discuss the quantity κ appearing in (2.15). For the

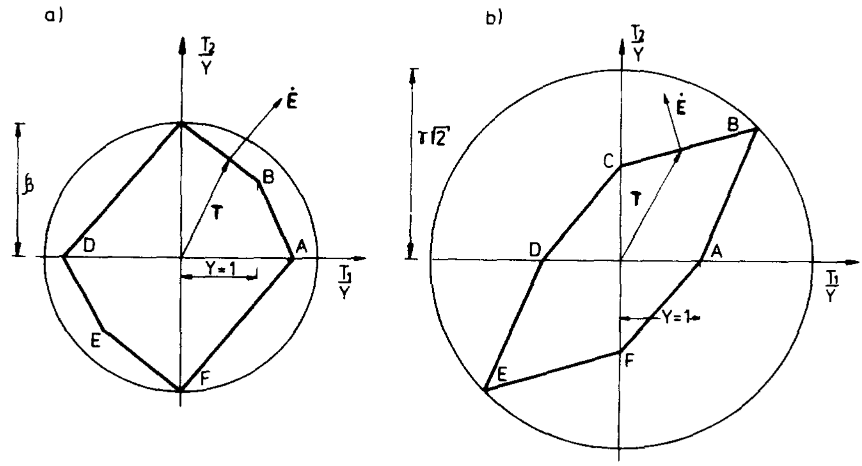

Fig. 2. Bounding of the internal dissipation depending the material orthotropies.

plane stress and the yield condition as shown in Fig. 2 eqn (2.16) leads to the following results

$$\kappa = \mathrm{Max}\,(\beta, \sqrt{2}, \gamma)\cdot Y \quad \text{or} \quad \kappa = \mathrm{Max}\,(\beta, \gamma\sqrt{2})\cdot Y \tag{3.4}$$

when we take Y_x as the reference yield stress Y and the respective formulas correspond to the situations shown in Fig. 2(a, b) consecutively. In comparison with the isotropic case discussed in [25] there is a difference in the value of κ when employing the principle of bounding for the internal dissipation. These values have to be taken into account when evaluating the estimate (2.19).

4. DEFLECTION ESTIMATES FOR ORTHOTROPIC CIRCULAR PLATES

Let us consider a rigid-plastic circular plate of cylindrical orthotropy such that $Y_r = \gamma Y$, $Y_\Theta = \beta Y$. The plate outer radius is A and its mass per unit surface of the middle surface is ρ. The plate will be subjected to loading which does not result in changing the directions of principal stress and strain rates, coinciding with those of orthotropy. The load is applied impulsively at $t = 0$ and the plate is subjected to motion which ceases at $t = t_f$, leaving a permanent deflection $W(t_f)$.

At moderately large deflections theory, if U denotes the radial displacement the equations of motion have the form

$$N_\Theta - (RN_r)' = -R\rho\ddot{U},$$

$$[-(RM_r)' + M_\Theta]' + (RN_rW')' + RP - \rho R\ddot{W} = 0. \tag{4.1}$$

where P stands for the surface vertical loading and the equations are written in a polar coordinate system R, Θ, t. The standard notation is used as regards differentiation with respect to the radial variable R and the time t.

The kinematical relations concerning the rates of strain $\dot{E}_r$, $\dot{E}_\Theta$ and curvature $\dot{K}_r$, $\dot{K}_\Theta$ of the middle surface are

$$\dot{E}_\Theta = \frac{\dot{U}}{R},\ \dot{E}_r = W'\dot{W}' + \dot{U}',\ \dot{K}_r = \dot{W}'',\ \dot{K}_\Theta = \frac{\dot{W}'}{R}. \tag{4.2}$$

To estimate deflections due to impact loading it is necessary to specify initial, terminal and boundary conditions. The initial conditions for the real deflections U, W are the following

$$W(R,0) = 0,\ \dot{W}(R,0) = \dot{W}_0,\ \dot{U}(R,0) = 0,\ U(R,0) = 0 \tag{4.3}$$

whereas the terminal conditions for the time instants t_f and t^* specified in Section 2 have the form

$$\dot{W}^*(R,t^*) = 0,\ \dot{U}^*(R,t^*) = 0,\ \dot{W}(R,t_f) = 0,\ \dot{U}(R,t_f) = 0. \tag{4.4}$$

The boundary conditions depend on the support requirements. For a simply supported plate they are

$$M_r(A,t) = 0,\ W(A,t) = \dot{W}(A,t) = 0,\ U(A,t) = \dot{U}(A,t) = 0. \tag{4.5}$$

The set of relations concerning bounding of deflections involves as well the plastic criterion as given in Table 1 and the plastic potential flow law relating the quantities appearing in (4.2) to the yield condition employed.

The principle (2.20) when applied to bounding the deflection W of a plate takes eventually the form

$$W_{\max} \geqslant \frac{\int_0^{t^*}\int_0^A P\dot{W}^*R\,dR\,dt + \int_0^A \rho\dot{W}_0\dot{W}_0^*R\,dR - \int_0^{t^*}\int_0^A D(\dot{\mathbf{K}}^*, \dot{\mathbf{E}}^*)R\,dR\,dt}{\int_0^A \rho R\ddot{W}^*(t^*)\,dR + \int_0^{t^*}\int_0^A \rho\dddot{W}^*R\,dR\,dt + C} \tag{4.6}$$

under the requirement that $\dot{U}^* = 0$.

In (4.6)

$$C = -\kappa A \int_0^{t^*} \dot{W}'^*\,dt\Big|_{R=A}. \tag{4.7}$$

This constant results from an estimation of the nonlinear term contribution to the internal energy disipation, as expressed in (2.19). Specifically the result follows when considering that W' and $\dot{W}'^*$ are of the same sign. Explicit inequalities used to derive (4.7) are

$$\int_0^{t^*}\int_0^A RN_r^{**}W'\dot{W}'^*\,dR\,dt \leqslant \kappa\int_0^{t^*}\int_0^A R|W'\dot{W}'^*|\,dR\,dt \leqslant$$

$$\leqslant -\kappa A\int_0^{t^*} \dot{W}'^*\Big|_{R=A} dt \cdot W_{\max} \tag{4.8}$$

where $W_{\max} = \text{Max}\, W(R, t)$, $\forall R, t \in (0, A)$, $(0, t_f)$. The derivation is analogous as for the isotropic case discussed in [16] and is not presented here. The only difference occurs in the value of κ, as specified in (3.4).

For computational reasons it appears useful to present (4.6) in the form.

$$W_{\max} \geqslant \frac{t^{*2}\int_0^1\int_0^1 P\dot{W}_r^*\,dr\,d\tau + t^*\int_0^1 \rho\dot{W}_0\dot{W}_0^*r\,dr - \frac{t^*M_0}{A^2}\int_0^1\int_0^1 D(\dot{\mathbf{K}}^*, \dot{\mathbf{E}}^*)r\,dr\,d\tau}{-\int_0^1 \rho\ddot{W}^*(1)r\,dr + \int_0^1\int_0^1 \rho\dddot{W}^*r\,dr\,d\tau + Ct^{*2}} \tag{4.9}$$

where

$$\tau = \frac{t}{t^*};\quad r = \frac{R}{A}.$$

The bound (4.9) is established assuming that W retains its modal form and is a monotonically increasing function of time and that $\partial W/\partial R$ has a constant sign within the range of spatial and time variables. In [16] the conditions imposed on W and $\dot{W}^*$ are specified. The conditions require that W' is continuous although formation of hinges is, in general, admissible. Also $\dot{W}^{*\prime\prime}$ is assumed negative within its range of continuity. Kinematically admissible velocity fields are taken of the form $\dot{U}^* = 0$, $\dot{W}^*(R, \tau) = V(\tau)a(r)$, $V(\tau)$ being a monotonously decreasing function of time.

Therefore $\dot{E}_r^* = 0$, $\dot{E}_\theta^* = 0$, $\dot{K}_r^* = \dot{W}_{,rr}^*$, $\dot{K}_\theta^* = \dot{W}_{,r}^*/A$. We shall not present (4.9) concerning the case involving displacement fields with hinges. The expressions for dissipation D are straightforward to obtain, but are algebraically complicated. In applications within this note we shall use only continuous and mode preserving velocity fields.

It is worthwhile to mention that the external energy introduced to the plate is related to the expression.

$$\int_0^1\int_0^1 P\dot{W}^*r\,dr\,d\tau + \int_0^1 \rho\dot{W}_0\dot{W}_0^*r\,dr. \tag{4.10}$$

For plates loaded by the velocity pulse the first integral vanishes whereas if a plate is loaded by the pressure pulse the second integral vanishes during the first stage of motion but has to be accounted for in the second stage when the load is removed but the motion slows down. The geometry changes influence on the permanent displacements is estimated by the constant C entering (4.9). For the limited interaction yield surface this constant is eventually obtained in the form

$$C = -\frac{\operatorname{Max}|N_r^{**}|}{A^2}\int_0^1 \dot{W}'^*\Big|_{r=1}\,\mathrm{d}\tau. \tag{4.11}$$

The maximization required in (4.11) is performed considering the hypersurface given in Table 1, thus (3.4) is used appropriately and eventually $\operatorname{Max}|N_r^{**}| = \gamma N_0$. If $\gamma = 1$, $\beta = 1$ the condition for isotropic plates is recovered. It should be mentioned again that the r.h.s. of (4.6) is maximized with respect to t^* so as to obtain the best lower bound.

As an example we consider a simply supported orthotropic plate loaded by the velocity pulse $\dot{W}_0 = \text{const.}$ over the entire surface. The boundary conditions of the problem are

$$\dot{W}(A,t) = 0,\ W(A,t) = 0,\ M_r(A,t) = 0. \tag{4.12}$$

For the yield criterion visualized in Fig. 1(a) a kinematically admissible velocity field is assumed in the following modal form

$$\dot{W}^* = V(t)(1 - R^{\beta}) \tag{4.13}$$

where the time independent term constitutes the solution of a static problem as derived in [22]. The velocity field does not contain plastic hinges. As regards $V(\tau)$ it is chosen to as $\dot{V} \leqslant 0$, $\ddot{V} \geqslant 0$, $\forall_\tau \in [0,1]$. The field selected satisfies all the conditions required by (4.9).

Calculations yield

$$W_{\max} \geqslant \frac{A^2\rho\dot{W}_0V_0t^* - 2t^{*2}(\beta+2)M_0\int_0^1 V(\tau)\,\mathrm{d}\tau}{-\rho\dot{V}_0A^2 + 2t^{*2}(\beta+2)N_0\int_0^1 V(\tau)\,\mathrm{d}\tau}. \tag{4.14}$$

where $V_0 = V(0)$, $\dot{V}_0 = \mathrm{d}V/\mathrm{d}\tau|_{\tau=0}$. The r.h.s of (4.14) attains its maximum for

$$t^* = \frac{H\dot{V}_0}{2\gamma V_0\dot{W}_0}\left[1 + \operatorname{Sgn}\dot{V}_0\sqrt{\left(1 - \frac{\rho A\dot{W}_0^2}{M_0H}\,\frac{\gamma V_0^2}{(\beta+2)\gamma N_0\int_0^1 V(\tau)\,\mathrm{d}\tau}\right)}\right] \tag{4.15}$$

Employing (4.15) in (4.14) one eventually obtains

$$W_{max} \geqslant \frac{H}{2\gamma}\,\frac{\chi[1 + \operatorname{Sgn}\dot{V}_0\sqrt{1+\chi}] + [1 + \operatorname{Sgn}\dot{V}_0\sqrt{1+\chi}]^2}{-\chi - [-1 + \operatorname{Sgn}\dot{V}_0\sqrt{1+\chi}]^2}. \tag{4.16}$$

where

$$\chi = \frac{\rho A^2\dot{W}_0^2}{M_0H}\cdot\frac{\gamma V_0^2}{(\beta+2)\dot{W}_0\int_0^1 V(\tau)\,\mathrm{d}\tau}. \tag{4.17}$$

Selecting now $V = (1-\tau)^n$, $n \geqslant 1$ one obtains

$$\operatorname{Sgn}\dot{V}_0 = -1;\quad \chi = \frac{\rho A^2\dot{W}_0^2}{M_0H}\times\frac{\gamma(n+1)}{n(\beta+2)}.$$

If now we consider an isotropic plate $\gamma = \beta = 1$ the following estimate is obtained

$$W_{\max} \geqslant \frac{H}{2}\,\frac{\dfrac{\rho\dot{W}_0^2A^2}{M_0H}\cdot\dfrac{n+1}{3n}\left[-1 + \sqrt{\left(1 + \dfrac{\rho\dot{W}_0^2A^2(n+1)}{3nM_0H}\right)}\right] - \left[-1 + \sqrt{\left(1 + \dfrac{\rho\dot{W}_0^2A^2(n+1)}{3nM_0H}\right)}\right]^2}{\dfrac{\rho\dot{W}_0^2A^2}{M_0H}\cdot\dfrac{n+1}{3n} + \left[-1 + \sqrt{\left(1 + \dfrac{\rho\dot{W}_0^2A^2(n+1)}{3nM_0H}\right)}\right]^2}. \tag{4.18}$$

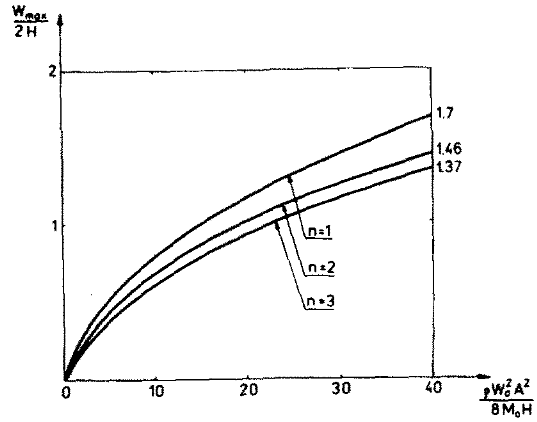

Fig. 3. Lower bounds to the permanent deflection at the plate center for various velocity fields.

The calculated estimates are traced in Fig. 3. It is seen that the best bound is for the considered velocity field is obtained at $n = 1$. Taking now

$$\dot W^* = (1-\varphi)(1-r^{\beta}) \tag{4.19}$$

the lower bound takes the form

$$\frac{W_{max}}{2H} \geqslant \frac{1}{4\gamma}\,\frac{\dfrac{2\rho\dot W_0^2A^2\gamma}{M_0H(\beta+2)}\left[-1+\sqrt{\left(1+\dfrac{2\rho A^2\dot W_0^2\gamma}{M_0H(\beta+\gamma)}\right)}\right]-\left[-1+\sqrt{\left(1+\dfrac{2\rho A^2\dot W_0^2\gamma}{M_0H(\beta+2)}\right)}\right]^2}{\dfrac{2\rho A^2\dot W_0^2\gamma}{M_0H(\beta+2)}+\left[-1+\sqrt{\left(1+\dfrac{2\rho A^2\dot W_0^2\gamma}{M_0H(\beta+2)}\right)}\right]^2}. \tag{4.20}$$

For various orthotropy ratios β the bound (4.20) is traced in Fig. 4. It is seen that the orthotropy influences the permanent deflections. Similar influence is observed for γ.

As a further example we consider the orthotropy shown in Fig. 1(c) taking the same loading and boundary conditions as before. Then the static solution obtained in [22] is assumed as a kinematically admissible field

$$\dot W^* = V(\tau)\left(1 - r^{\beta/\gamma}\right), \tag{4.21}$$

where $V(\tau)$ is such that $\dot V \leqslant 0$, $\ddot V \geqslant 0\ \forall \tau \in [0,1]$. This field can be employed in (4.14). The linear part of the strain rate tensor is

$$\dot E^*_{\vartheta} = 0;\ \dot E^*_r = 0;\ \dot K^*_{\vartheta} = -V(\tau)\frac{\beta}{\gamma}r^{\beta/\gamma-2};\ -\dot K^*_r = V(\tau)\frac{\beta}{\gamma}\left(\frac{\beta-\gamma}{\gamma}\right)r^{\beta/\gamma-2}. \tag{4.22}$$

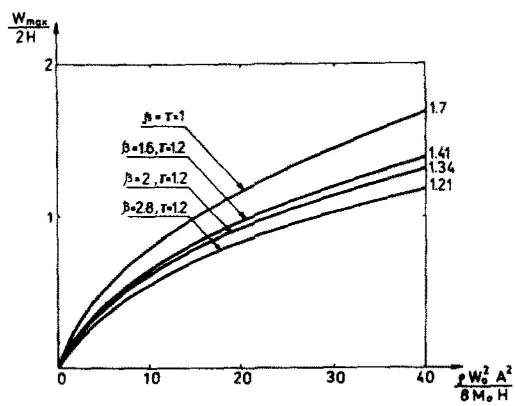

Fig. 4. Lower bounds to the permanent deflection at the plate center depending on the plastic orthotropy.

During the motion the yield condition is satisfied within the stress profile BC in Fig. 1(c). If $\beta > \gamma$ there is a possibility of plastic hinge formation. Such a case will not be considered here. The dissipation function of the linear part is

$$D(\dot{\mathbf{E}}^*, \dot{\mathbf{K}}^*) = M_0 V(\tau) \frac{\beta^2}{\gamma} r^{\beta/\gamma - 2}. \tag{4.23}$$

Eventually the kinematically admissible field (4.21) results in the lower bound to the permanent maximum deflection

$$W_{max} \geqslant \frac{\rho \dot{W}_0 V_0 A^2 t^* - 2M_0 t^{*2}(\beta + 2\gamma) \int_0^1 V(\tau)\,d\tau}{-\rho \dot{V}_0 A^2 + 2N_0(\beta + 2\gamma) \int_0^1 V(\tau)\,d\tau \cdot t^{*2}}. \tag{4.24}$$

The r.h.s. attains its maximum at

$$t^* = \frac{H}{2} \cdot \frac{\dot{V}_0}{V_0 \dot{W}_0} \cdot [1 + \mathrm{Sgn}\ \dot{V}_0 \sqrt{1 - \chi}] \tag{4.25}$$

where

$$\chi = \frac{\rho \dot{W}_0^2 A^2}{M_0 H} \cdot \frac{V_0^2}{\dot{V}_0(\beta + 2\gamma) \int_0^1 V(\tau)\,d\tau}. \tag{4.26}$$

at $V_0 = V(0)$, $\dot{V}_0 = \dot{V}(0)$. Introducing (4.25) into (4.24) the following lower bound to the permanent deflection after impact is obtained

$$W_{max} \geqslant \frac{H}{2} \frac{\chi[1 + \mathrm{Sgn}\ \dot{V}_0\sqrt{1-\chi}] - [1 + \mathrm{Sgn}\ \dot{V}_0\sqrt{1-\chi}]^2}{-\chi + [1 + \mathrm{Sgn}\ \dot{V}_0\sqrt{1-\chi}]^2}. \tag{4.27}$$

If, moreover $V(\tau) = 1 - \tau$ is chosen the bound is

$$W_{max} \geqslant \frac{H}{2} \frac{\frac{2\rho A^2 \dot{W}_0^2}{M_0 H(\beta + 2\gamma)}\left[-1 + \sqrt{\left(1 + \frac{2\rho A^2 \dot{W}_0^2}{M_0 H(\beta + 2\gamma)}\right)}\right] - \left[-1 + \sqrt{\left(1 + \frac{2\rho A^2 \dot{W}_0^2}{M_0 H(\beta + 2\gamma)}\right)}\right]^2}{\frac{2\rho A^2 \dot{W}_0^2}{M_0 H(\beta + 2\tau)} + \left[-1 + \sqrt{\left(1 + \frac{2\rho A^2 \dot{W}_0^2}{M_0 H(\beta + 2\gamma)}\right)}\right]}. \tag{4.28}$$

The results are plotted in Fig. 5 where, in addition, a comparison is made with the bound obtained employing the linear plate theory. It is seen that significant differences in the permanent displacements are observed in comparison with the linear theory, represented by a straight line OA. In the considered case of orthotropy the permanent deflections are smaller than in the isotropic case. The solutions regarding isotropy can be found in [16]. The essential difference in the case of orthotropy is that the velocity field does not make a developable cone but is of the form (4.13) or (4.21), as derived in [18] for the static case.

5. ORTHOTROPIC CYLINDRICAL SHELL

As a further example of application of the lower bound estimate (2.20) to the permanent deflections due to instantaneous loading we consider a cylindrical sandwich shell of radius A and length $2L$. The shell is orthotropic and the yield properties in the axial and circumferential directions are Y_x, Y_φ respectively. A cylindrical system of coordinates is chosen, the stress resultants being N_x, M_x, N_φ, M_φ. The radial and axial displacements are W and U appropriately. The loading is such that the principal directions of stress and strain do not vary and coincide

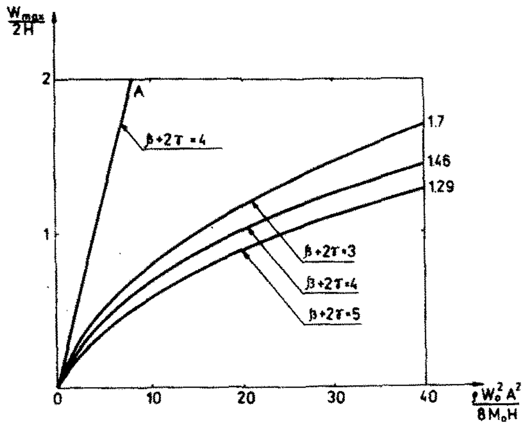

Fig. 5. Comparision of permanent deflections after pulse loading according to the linear and the moderately large displacement theories.

with the privileged directions of orthotropy. The origin of the coordinate system is in the middle of shell length.

The following dimensionless quantities will be employed if not stated otherwise

$$x=\frac{X}{L};\ c=\frac{L^2}{AH},\ \tau=\frac{t}{t^*}$$

$$n_x=\frac{N_x}{N_0},\ n_\varphi=\frac{N_\varphi}{N_0},\ m_\varphi=\frac{M_\varphi}{M_0},\ m_x=\frac{M_x}{M_0} \tag{5.1}$$

$N_0=2Yh$, $M_0=2YHh$, Y being a reference yield stress and h denotes the plastic layer thickness of the sandwich shell of thickness $2H$.

The equations of motion under the assumptions of moderately large deflections under rotationally symmetric loading are

$$N_x'-\rho\ddot{U}=0,\ Q_x=-M_x',$$

$$-M_x''+(N_xW')'-\frac{N\varphi}{A}-\rho\ddot{W}=0 \tag{5.2}$$

where the only independent variable is the axial coordinate.

The kinematical relations result in the following expressions for the extension and curvature rates

$$\dot{\lambda}_x=\dot{U}'+W'\dot{W}',\ \dot{\lambda}_\varphi=\frac{\dot{W}}{A},\ \dot{\kappa}_x=\dot{W}'',\ \dot{\kappa}_\varphi=0. \tag{5.3}$$

Due to symmetry, the boundary, initial and final conditions, written for $0\leqslant X\leqslant L$ are

$$W(L,t)=\dot{W}(L,t)=0,\ \dot{W}^*(L,t)=0 \tag{5.4}$$

$$U(x,0)=W(x,0)=0,\ \dot{U}(x,0)=0,\ \dot{W}(x,0)=\dot{W}_0 \tag{5.5}$$

$$\dot{U}(x,t_f)=\dot{W}(x,t_f)=0,\ \dot{U}^*(x,t^*)=\dot{W}^*(x,t^*)=0 \tag{5.6}$$

where the respective constraints at the ends depend upon the conditions of support. The requirements (5.4) and (5.6) have to be accounted for when selecting a kinematically admissible displacement rates $\dot{U}^*$ and $\dot{W}^*$.

For the yield surface shown in Table 2 the following ratios were employed regarding orthotropy

$$\beta = \frac{Y_x}{Y};\ \gamma = \frac{Y_\varphi}{Y};\ \beta \geq \frac{\gamma}{\gamma + 1}. \tag{5.7}$$

The table concerns eventually the case when $\beta = 1$ thus the yield stress in the axial direction is chosen for reference. As $\dot{\kappa}_\varphi = 0$ the circumferential moment can be eliminated from the yield criterion and hence the results given in Table 2 concern explicitly infinitesimal motion of such a moderately displacements when $\dot{\kappa}_\varphi = 0$ is justified.

The recalled principal equations of plastic shells at moderately large deflections allow to pass to specifications of the bounding principle (2.20). We require further that the real deflection W preserves its mode, is a monotonic function of time and that W' has a constant sign for $\forall x \in [0, L]$, $\forall t \in [0, t_f]$. The bounding principle takes the form

$$W_{\max} \geq \frac{t^* \int_0^1 \int_0^1 P\dot{W}^* \, dx \, d\tau + \int_0^1 \rho \dot{W}_0 \dot{W}_0^* \, dx - t^* \int_0^1 \int_0^1 D(\dot{\mathbf{E}}^*, \dot{\mathbf{K}}^*) \, dx \, d\tau}{-\int_0^1 \frac{\dot{W}^*(1)}{t^*} dx + \frac{1}{t^*} \int_0^1 \int_0^1 \rho \dddot{W}^* \, dx \, d\tau + Ct^*} \tag{5.8}$$

where t^*, is as before, the terminal time of motion according to the kinematically admissible deflection rate $\dot{U}^* = 0$, $\dot{W}^* \neq 0$. A kinematically admissible strain rates are

$$\dot{\lambda}_x^* = 0, \dot{\lambda}_\varphi^* = \frac{\dot{W}^*}{A};\ \dot{\kappa}_x^* = \frac{\dot{W}_{,xx}^*}{A} \cdot \frac{AH}{2L}, \dot{\kappa}_\varphi^* = 0. \tag{5.9}$$

Making use of these quantities in the expression for the dissipation $D(\dot{\mathbf{E}}^*, \dot{\mathbf{K}}^*)$ entering the numerator in (5.8) we obtain for a simply supported shell the expression

$$\int_0^1 \int_0^1 D(\dot{\mathbf{E}}^*, \dot{\mathbf{K}}^*) \, dx \, d\tau = \frac{2M_0}{AH} \sum_k \int_{x_k}^{x_{k+1}} \int_0^1 \left(n_\varphi^* \dot{W}^* + m_x^* \frac{\dot{W}_{,xx}^*}{c^2} \right) dx \, d\tau$$
$$- \frac{M_0}{L} \sum_k \int_0^1 m_x^* \Big|_{x_k} [\dot{W}_{,x}^*]_{x_k} \, d\tau \tag{5.10}$$

where the internal forces in the field and at the hinges are obtained from the yield criterion employing the plastic potential flow law and imposing the prescribed motion (5.9). Usually we shall use continuous kinematically admissible fields so that the terms concerning radial hinges disappear except possibly the case of a clamped shell.

The constant C appearing in the denominator of (5.8) can eventually be written in the form

$$C = \operatorname{Max} |N_x^{**}| \frac{1}{L^2} \int_0^1 \dot{W}_{,x} \Big|_{x=1} d\tau. \tag{5.11}$$

The maximization has to be done employing the yield condition as given for example in Table 2. In this case $\operatorname{Max} |N_x^{**}| = N_0$. The time t^* is obtained similarly as in the previously discussed case of plates requiring that the r.h.s. of (5.8) attains its maximum.

As an example let us consider a simply supported shell loaded by the pulse of velocity $\dot{W}_0 = \text{const}$. The boundary conditions are

$$\dot{W}(L, t) = W(L, t) = 0,\ M_x(L, t) = 0, \quad \forall t \in [0, t_f] \tag{5.12}$$

Except for a time dependent multiplier, we choose a kinematically admissible displacement rate field as in the static case of orthotropic shell,[22]

$$\dot{W}^* = V(\tau) Sh\left[\sqrt{\frac{2-\gamma}{\gamma}}\, c(1-x)\right],\ \dot{U}^* = 0 \tag{5.13}$$

where $V(\tau)$ is to be selected so as to fulfil the conditions $\dot{V}(1) \leqslant 0$, $\ddot{V}(\tau) \geqslant 0$, $\forall \tau \in [0, 1]$. The field (5.13) satisfies all the requirements imposed on kinematically admissible fields suitable to considerations when employing (5.8). The linear part of the strain rate tensor is

$$\dot{\kappa}_{\varphi}^{*} = 0, \dot{\lambda}_{x}^{*} = 0, \dot{\lambda}_{\varphi}^{*} = \frac{V(\tau)}{A} Sh\left[\sqrt{\frac{2-\gamma}{\gamma}}\, c(1-x)\right], \dot{\kappa}_{x}^{*} = \frac{V(\tau)}{A} \frac{2-\gamma}{\gamma} Sh\left[\sqrt{\frac{2-\gamma}{\gamma}}\, c(1-x)\right]. \tag{5.14}$$

Employing (5.13) in (5.8) one eventually obtains

$$W_{\max} \geqslant \frac{\left[\rho \dot{W}_0 V(0) - \frac{2M_0 t^*}{\gamma AH} \int_0^1 V(\tau)\, d\tau\right] \int_0^1 Sh\left[\sqrt{\frac{2-\gamma}{\gamma}}\, cx\right] dx}{-\rho \frac{\dot{V}(0)}{t^*} \int_0^1 Sh\left[\sqrt{\frac{2-\gamma}{\gamma}}\, cx\right) dx + \frac{N_0 c t^*}{L^2} \int_0^1 V(\tau)\, d\tau}. \tag{5.15}$$

Imposing a particular form of $V(\tau)$, on the basis of considerations concerning plates, we assume

$$V(\tau) = (1-\tau). \tag{5.16}$$

Then (5.15) results in

$$W_{\max} \geqslant \left[Ch\sqrt{\frac{2-\gamma}{\gamma}\, c^2} - 1\right] \cdot \frac{\rho \dot{W}_0 t^* - \frac{M_0 t^{*2}}{\gamma AH}}{\rho\left[Ch\sqrt{\frac{2-\gamma}{\gamma}\, c^2} - 1\right] + \frac{2-\gamma}{\gamma} \frac{N_0 c^2}{2L^2} t^{*2}}. \tag{5.17}$$

The r.h.s. attains its maximum for

$$t^* = H \cdot \frac{Ch\sqrt{\frac{2-\gamma}{\gamma}\, c^2} - 1}{(2-\gamma)\dot{W}_0} \left[-1 + \left(1 + \frac{2\rho \dot{W}_0^2 L^2}{N_0 H} \cdot \frac{\gamma(2-\gamma)}{c^2\left[Ch\sqrt{\frac{2-\gamma}{\gamma}\, c^2} - 1\right]}\right)\right]. \tag{5.18}$$

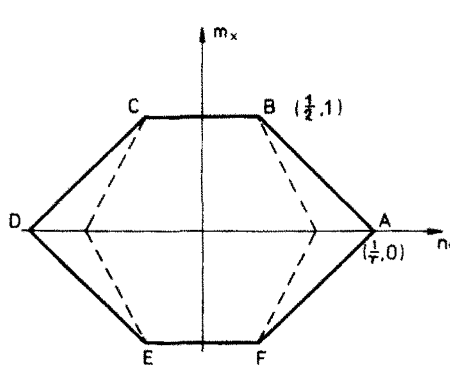

Fig. 6.

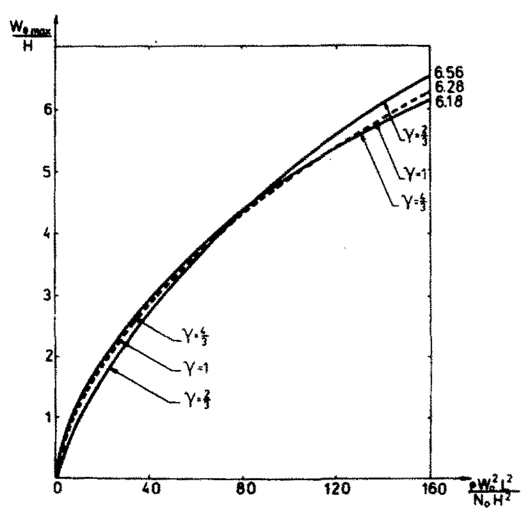

Fig. 7.

Fig. 6. Orthotropic yield criterion for cylindrical shells.

Fig. 7. Lower bounds to the permanent deflection at the center of orthotropic shells at various ratios of the circumferential to axial yield points.

Eventually from (5.17) and (5.18) a lower bound to the maximum permanent deflection is obtained in the form

$$W_{\max} \geqslant \frac{\left[Ch\sqrt{\frac{2-\gamma}{\gamma}c^2} - 1\right]}{2-\gamma} \cdot \frac{\chi[-1+\sqrt{1+\chi}] - [-1+\sqrt{1+\chi}]^2}{\chi + [-1+\sqrt{1+\chi}]^2} \quad (5.19)$$

where

$$\chi = \frac{2\rho \dot{W}_0^2 L^2}{N_0 H^2} \cdot \frac{\gamma(2-\gamma)}{c^2\left(Ch\sqrt{\frac{2-\gamma}{\gamma}c^2} - 1\right)}. \quad (5.20)$$

For the particular case $c^2 = 1$ the bound (5.19) is shown in Fig. 7. The result is compared with the isotropic case when $\gamma = 1$. It is seen that the final deflection is nonlinear with respect to the initial velocity and how orthotropy influence the deformed shape. To make conclusive remarks as regards the influence of orthotropy various kinematically admissible velocity fields $\dot{W}^*$ should be considered allowing, possibly, for the collapse modes involving hinges. Such a possibility is included in the general formula (5.8).

6. CONCLUSIONS

The method concerning estimation from below of the permanent deflections of dynamically loaded structures and allowing to take into account the influence of geometrical nonlinearity on the final response can be extended on orthotropic solids in a straightforward manner. The method developed in [16, 17] allows to bound from below the permanent deflections of impacted structures considered in [20] but now at their moderately large deflection. The method employs the equation of motion referred to the undeformed configuration thus the deformed shape enters the basic relations. The unknown displacements are estimated via introducing a kinematically admissible velocity field $\dot{U}_i^*(x, t) = V(t)A(x)$, thus assuming specific modes of deformation. As the modes of deformation those concerning the static response can be selected and optimalization of the time function $V(t)$ is usually made. The general expression for a lower bound estimate involves the time of motion t^*. This quantity can be evaluated employing the standard extremum condition considering the actually chosen kinematically admissible velocity field $\dot{U}_i^*$. Restrictions regarding smoothness of the fields involved were applied in the considered examples but the general expressions allow for the deflection modes with hinges, expressions for the energy dissipation then being more cumbersome in application.

Accounting for the plastic anisotropy modifies the expressions concerning the internal dissipation and therefore an estimate of the nonlinear part of the dissipation, in addition to influencing values of the linear part.

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